JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **201,** 226–246 (1996)
ARTICLE NO. 0252

Hyperplane Means of Potentials

Yoshihiro Mizuta

Department of Mathematics, Faculty of Integrated Arts and Sciences, Hiroshima University, Higashi-Hiroshima 739, Japan

Submitted by E. Stanley Lee

etadata, citation and similar papers at core.ac.uk

1. INTRODUCTION

For $0 < \alpha < n$, we define the Riesz potential

$$U_\alpha f(x) = \int |x-y|^{\alpha-n} f(y)\, dy$$

for a measurable function f on R^n satisfying

$$\int_{R^n} (1+|y|)^{\alpha-n} |f(y)|\, dy < \infty \tag{1.1}$$

and

$$\int_{R^n} |f(y)|^p |y_n|^\beta\, dy < \infty, \qquad y = (y_1, \ldots, y_n). \tag{1.2}$$

Note here that (1.1) is equivalent to the condition that

$$U_\alpha |f| \not\equiv \infty. \tag{1.3}$$

We first study the limiting behavior of the qth hyperplane means of $U_\alpha f$ over the surfaces $D(r) = \{x = (x', x_n) \in R^{n-1} \times R^1 \colon x_n = r\}$; in fact we show that

$$\liminf_{r \to 0} r^{(n-\alpha p+\beta)/p-(n-1)/q} S_q(u_r) = 0$$

for suitable q satisfying $(n-\alpha p+\beta)/p > (n-1)/q$, where $u_r(x') = U_\alpha f(x', r)$ and

$$S_q(u_r) = \left(\int_{R^{n-1}} |u_r(x')|^q\, dx' \right)^{1/q}.$$

0022-247X/96 $18.00
Copyright © 1996 by Academic Press, Inc.
All rights of reproduction in any form reserved.

In case $(n - \alpha p + \beta)/p = (n-1)/q$, we apply Sobolev's inequality to show that $S_q(u_r)$ is bounded. Next we consider the existence of

$$\lim_{r \to 0} r^{(n - \alpha p + \beta)/p - (n-1)/q} S_q(v_r) = 0,$$

where $(n - \alpha p + \beta)/p < (n-1)/q$ and $v_r(x') = U_\alpha f(x', r) - U_\alpha f(x', 0)$.

Similar results can be obtained for p-precise functions u on the half space $D = \{x = (x', x_n) \in R^{n-1} \times R^1 \colon x_n > 0\}$ such that

$$\int_D |\operatorname{grad} u(x)|^p |x_n|^\beta \, dx < \infty; \tag{1.4}$$

for p-precise functions, see Ohtsuka [12] and Ziemer [16]. For this purpose we may apply the integral representations given by [3, 8, 9]. We are concerned only with harmonic functions on D satisfying (1.4). Professor Yamashita [15, Theorem 3] showed that if u is a harmonic function on the unit ball B satisfying

$$\int_B |\operatorname{grad} u(x)|^2 \left(1 - |x|^2\right)^\beta dx < \infty, \qquad 0 \leqq \beta \leqq 1,$$

then $S_q^*(u_r)$ is bounded on $(0, 1)$ for $1/q = (n - 2 + \beta)/2(n-1) > 0$, where

$$S_q^*(u_r) = \left(\frac{1}{\omega_{n-1}} \int_{\partial B} |u(r\xi)|^q \, dS(\xi) \right)^{1/q}$$

with ω_{n-1} denoting the area of ∂B. We are able to give a generalization of his result only in case $\beta < p - 1$; the case $\beta = p - 1$ remains open.

There are several results for Green potentials; see Gardiner [1], Mizuta [10], and Stoll [13, 14]. For local growth properties, we refer to Gardiner [2] and Mizuta [4, 5, 7, 11].

2. HYPERPLANE MEANS OF POTENTIALS

In this paper, let $M, M_1, M_2, \ldots,$ denote various constants independent of the variables in question.

Set

$$R_\alpha(x, y) = |x - y|^{\alpha - n}, \qquad 0 < \alpha < n.$$

LEMMA 2.1. *If* $-1 < \beta < 0$ *and* $0 < (\alpha - n)q + n < -\beta$, *then*

$$\left(\int R_\alpha(x, y)^q |y_n|^\beta \, dy\right)^{1/q} \leqq M x_n^{\alpha - n + (n+\beta)/q}$$

for any $x = (x', x_n) \in D$, *where* M *is a positive constant independent of* x.

Proof. Consider the sets

$$E_1 = \{y = (y', y_n) : y_n \geqq x_n/2\},$$
$$E_2 = \{y = (y', y_n) : y_n < x_n/2\}.$$

If $y \in E_1$, then $x_n + |x_n - y_n| \leqq 3|y_n|$. Hence, applying the polar coordinates about x, we have

$$\begin{aligned}
&\left(\int_{E_1} R_\alpha(x, y)^q |y_n|^\beta \, dy\right)^{1/q} \\
&\qquad \leqq M_1 \left(\int_{E_1} |x - y|^{q(\alpha - n)} (|x_n| + |y_n - x_n|)^\beta \, dy\right)^{1/q} \\
&\qquad \leqq M_2 \left(\int_{x_n}^\infty r^{q(\alpha - n) + \beta} r^{n-1} \, dr\right)^{1/q} + M_2 x_n^{\beta/q} \left(\int_0^{x_n} r^{q(\alpha - n)} r^{n-1} \, dr\right)^{1/q} \\
&\qquad \leqq M_3 x_n^{\alpha - n + (n+\beta)/q}.
\end{aligned}$$

If $y \in E_2$, then $|x_n| + |y_n| \leqq 3|x_n - y_n|$, so that, as above, we have

$$\begin{aligned}
\left(\int_{E_2} R_\alpha(x, y)^q |y_n|^\beta \, dy\right)^{1/q} &\leqq M_4 \left(\int_{E_2} (|x' - y| + x_n)^{q(\alpha - n)} |y_n|^\beta \, dy\right)^{1/q} \\
&\leqq M_5 \left(\int_0^\infty (r + x_n)^{q(\alpha - n)} r^\beta r^{n-1} \, dr\right)^{1/q} \\
&\leqq M_6 x_n^{\alpha - n + (\beta + n)/q},
\end{aligned}$$

where x' is identified with the perpendicular projection $(x', 0)$ of x.

LEMMA 2.2. *Let* $\alpha - n + (n-1)/q < 0$. *For* $x = (x', x_n)$ *and* $y = (y', y_n)$ *in* R^n,

$$\left(\int_{R^{n-1}} R_\alpha(x, y)^q \, dx'\right)^{1/q} \leqq M |y_n - x_n|^{\alpha - n + (n-1)/q}.$$

Proof. For $x = (x', x_n)$ and $y = (y', y_n)$, set

$$r = |x' - y'|, \qquad a = |y_n - x_n|.$$

Then

$$\left(\int_{R^{n-1}} R_\alpha(x, y)^q \, dx' \right)^{1/q} = M_1 \left(\int_0^\infty (r^2 + a^2)^{q(\alpha - n)/2} r^{n-2} \, dr \right)^{1/q}$$
$$= M_2 a^{\alpha - n + (n-1)/q}.$$

We need the following result (cf. [10, Corollary to Lemma 2]).

LEMMA 2.3. *Let $0 < d < 1$ and μ be a finite measure on R^1. Then*

$$\liminf_{t \to 0, t > 0} t^d \int_{R^1} |t - s|^{-d} \, d\mu(s) = \mu(\{0\}).$$

THEOREM 2.1. *Let $\beta < p - 1, 1 \leqq p \leqq q$, and*

$$\frac{n - \alpha p - 1}{p(n-1)} < \frac{1}{q} < \frac{n - \alpha p + \beta}{p(n-1)}. \tag{2.1}$$

If f is a nonnegative measurable function on R^n satisfying (1.2), *then*

$$\liminf_{r \to 0} r^{(n - \alpha p + \beta)/p - (n-1)/q} S_q(u_r) = 0,$$

where $u_r(x') = U_\alpha f(x', r)$ for $r > 0$.

Proof. First note that $p > 1$ and

$$\frac{1}{q} < \frac{n - \alpha p + \beta}{p(n-1)} < \frac{n - \alpha}{n - 1}.$$

Take (δ, γ) such that

$$\beta < \gamma < p - 1, \qquad 0 < \delta < 1,$$
$$p(n - \alpha)\delta + \alpha p - n > 0, \tag{2.2}$$
$$p(n - \alpha)\delta + \alpha p - n < \gamma < p(n - \alpha)\delta + \beta - p(n-1)/q, \tag{2.3}$$

and

$$\frac{n-1}{q(n-\alpha)} < \delta < \frac{n-1}{q(n-\alpha)} + \frac{1}{p(n-\alpha)}. \tag{2.4}$$

Set $a = (1-\delta)p^*$ and $b = -\gamma p^*/p$, where $1/p + 1/p^* = 1$. Then it follows from (2.2) that $\alpha > n/a^*$, where $1/a + 1/a^* = 1$. Further, note that $-1 < b < 0$ and by (2.3),

$$\frac{b}{a} < \frac{n}{a^*} - \alpha < 0.$$

Using Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned} &\left|u_{x_n}(x')\right| \\ &\quad \leqq \left(\int R_\alpha(x,y)^{(1-\delta)p^*}|y_n|^{-\gamma p^*/p}\,dy\right)^{1/p^*}\left(\int R_\alpha(x,y)^{\delta p}|y_n|^\gamma f(y)^p\,dy\right)^{1/p} \\ &\quad \leqq M_1 x_n^{(\alpha-n)(1-\delta)+n/p^*-\gamma/p}\left(\int R_\alpha(x,y)^{\delta p}|y_n|^\gamma f(y)^p\,dy\right)^{1/p}. \end{aligned}$$

In view of Minkowski's inequality we obtain

$$\begin{aligned} S_q(u_{x_n}) &\leqq M_1 x_n^{(\alpha-n)(1-\delta)+n/p^*-\gamma/p} \\ &\qquad \times\left\{\int\left(\int_{R^{n-1}} R_\alpha(x,y)^{\delta q}\,dx'\right)^{p/q}|y_n|^\gamma f(y)^p\,dy\right\}^{1/p}. \end{aligned}$$

Since $\alpha - n + (n-1)/(\delta q) < 0$ by (2.4), we have by Lemma 2.2

$$\left(\int_{R^{n-1}} R_\alpha(x,y)^{\delta q}\,dx'\right)^{p/q} \leqq M_2\left[|y_n - x_n|^{\alpha-n+(n-1)/\delta q}\right]^{\delta p}.$$

Consequently,

$$\begin{aligned} S_q(u_r) &\leqq M_3 r^{(\alpha-n)(1-\delta)+n/p^*-\gamma/p} \\ &\qquad \times\left(\int\left\{\left[|y_n - r|^{\alpha-n+(n-1)/\delta q}\right]^{\delta p}|y_n|^{\gamma-\beta}\right\}|y_n|^\beta f(y)^p\,dy\right)^{1/p}. \end{aligned}$$

For simplicity, set

$$d = -[\alpha - n + (n-1)/\delta q](\delta p).$$

Then $0 < d < 1$ by (2.4). Consider

$$\begin{aligned} K(r, y_n) &= r^{p[(n-\alpha p+\beta)/p-(n-1)/q]} r^{p[(\alpha-n)(1-\delta)+n/p^*-\gamma/p]} \\ &\qquad \times |r - y_n|^{[\alpha-n+(n-1)/\delta q]\delta p}|y_n|^{\gamma-\beta}. \end{aligned}$$

In view of the right inequality of (2.3), we see that

$$r^{(n-\alpha p+\beta)/p-(n-1)/q} r^{(\alpha-n)(1-\delta)+n/p^*-\gamma/p} = r^{(n-\alpha)\delta+(\beta-\gamma)/p-(n-1)/q} \to 0$$

as $r \to 0$. Further, if $y_n < 0$ or $3r/2 < y_n$, then

$$K(r, y_n) \leqq M_4 \big(r/|y_n|\big)^{(n-\alpha)\delta p+(\beta-\gamma)-p(n-1)/q} \leqq M_5;$$

if $|y_n| < r/2$, then we also have

$$K(r, y_n) \leqq M_6.$$

If $|r - y_n| < r/2$, then

$$K(r, y_n) \leqq M_7 r^d |r - y_n|^{-d}.$$

Now applying Lemma 2.3 with $d\mu(y_n) = |y_n|^{\beta}(\int_{R^{n-1}} f(y)^p \, dy')\, dy_n$, we insist that

$$\liminf_{r\to 0} r^{(n-\alpha p+\beta)/p-(n-1)/q} S_q(u_r) = 0.$$

THEOREM 2.2. *Let* $p > 1$, $0 \leqq \beta < p\min\{\alpha, 1\} - 1$, *and*

$$\frac{1}{q} = \frac{n - \alpha p + \beta}{p(n-1)} > 0.$$

If f is a nonnegative measurable function on R^n satisfying (1.2), *then*

$$S_q(u_r) \leqq M\left(\int f(y)^p |y_n|^{\beta}\, dy\right)^{1/p}$$

for any $r > 0$.

Proof. Applying Hölder's inequality and the calculations as in Lemma 2.1, we have

$$\left|u_{x_n}(x')\right| \leqq \int_{R^{n-1}} \left(\int R_\alpha(x,y)^{p^*} |y_n|^{-\beta p^*/p}\, dy_n\right)^{1/p^*} \left(\int |y_n|^{\beta} f(y)^p \, dy_n\right)^{1/p} dy'$$

$$\leqq M_1 \int_{R^{n-1}} |x' - y'|^{\alpha-n+1/p^*-\beta/p} \left(\int |y_n|^{\beta} f(y)^p\, dy_n\right)^{1/p} dy'.$$

If $1/q = 1/p - (\alpha - 1/p - \beta/p)/(n-1)$, then we apply Sobolev's inequality to establish the required inequality.

Remark 2.1. In case $-1 < \beta < p - 1$ and $n - \alpha p + \beta > 0$, (1.2) implies (1.1) and thus (1.3), by Hölder's inequality.

3. BOUNDARY LIMITS IN S_q-NORM

Consider the function

$$K_\alpha(x,y) = |x-y|^{\alpha-n} - |x'-y|^{\alpha-n},$$

where x' is identified with the perpendicular projection $(x',0)$ of $x = (x', x_n)$. Set

$$E(x_n) = \{y = (y', y_n)\colon y_n > x_n/2\},$$
$$F(x_n) = \{y = (y', y_n)\colon y_n < x_n/2\}.$$

LEMMA 3.1 (cf. [6, Lemma 1]). *If $x = (x', x_n) \in D$ and $y \in E(x_n)$, then*

$$|K_\alpha(x,y)| \leqq Mx_n(2y_n - x_n)|x-y|^{\alpha-n}|x'-y|^{-2};$$

if $x = (x', x_n) \in D$ and $y \in F(x_n)$, then

$$|K_\alpha(x,y)| \leqq Mx_n(x_n - 2y_n)|x'-y|^{\alpha-n}|x-y|^{-2}.$$

For a point $x \in R^n$ and $r > 0$, we denote by $B(x,r)$ the open ball with center at x and radius r.

LEMMA 3.2. *Let $\beta > -1$, $\alpha q^* > n$, and*

$$\frac{n}{q^*} - \alpha < \frac{\beta}{q} < \frac{n}{q^*} - \alpha + 1, \qquad \frac{1}{q} + \frac{1}{q^*} = 1.$$

Then

$$\left(\int |K_\alpha(x,y)|^q |y_n|^\beta\, dy\right)^{1/q} \leqq Mx_n^{\alpha-n+(n+\beta)/q}$$

for any $x = (x', x_n) \in D$, where M is a positive constant independent of x.

Proof. Consider the sets

$$E_1 = \{y = (y', y_n)\colon y_n \geqq 2x_n\},$$
$$E_2 = \{y = (y', y_n)\colon x_n/4 < y_n < 2x_n\},$$
$$E_3 = \{y = (y', y_n)\colon y_n \leqq x_n/4\}.$$

By Lemma 3.1, if $y \in E_1$, then

$$|K_\alpha(x,y)| \leqq 3Mx_n|x-y|^{\alpha-n-1}, \qquad y_n - x_n < y_n < 2(y_n - x_n).$$

Since $\alpha - n - 1 + (n+\beta)/q < 0$, applying the polar coordinates about x, we have

$$\begin{aligned}\left(\int_{E_1} K_\alpha(x,y)^q |y_n|^\beta\, dy\right)^{1/q} &\leqq M_1 x_n \left(\int_{E_1} \left[|x-y|^{\alpha-n-1}\right]^q (y_n - x_n)^\beta\, dy\right)^{1/q} \\ &\leqq M_2 x_n \left(\int_{x_n}^{\infty} r^{q(\alpha-n-1)+\beta} r^{n-1}\, dr\right)^{1/q} \\ &\leqq M_3 x_n^{\alpha-n+(n+\beta)/q}.\end{aligned}$$

If $\beta \leqq 0$, then, by Lemma 3.1, we have

$$\begin{aligned}&\left(\int_{E_2} |K_\alpha(x,y)|^q |y_n|^\beta\, dy\right)^{1/q} \\ &\quad\leqq M_4 x_n^{\beta/q} \left(\int_{B(x,x_n/4)} |x-y|^{q(\alpha-n)}\, dy\right)^{1/q} \\ &\qquad + M_4 x_n^{\beta/q+(\alpha-n)} \left(\int_{B(x'',x_n/4)} dy\right)^{1/q} \\ &\qquad + M_4 x_n \left(\int_{E_2 - B(x'',x_n/4)} |x''-y|^{(\alpha-n-1)q} |y_n - x_n/2|^\beta\, dy\right)^{1/q} \\ &\quad\leqq M_5 x_n^{\beta/q+\alpha-n+n/q},\end{aligned}$$

where $x'' = (x+x')/2 = (x', x_n/2)$; if $\beta > 0$, then, since $0 < \alpha - n + n/q < 1$, we have

$$\begin{aligned}&\left(\int_{E_2} |K_\alpha(x,y)|^q |y_n|^\beta\, dy\right)^{1/q} \\ &\quad\leqq M_6 x_n^{\beta/q} \left(\int_{B(x,x_n/4)} |x-y|^{q(\alpha-n)}\, dy\right)^{1/q} \\ &\qquad + M_6 x_n^{\beta/q+(\alpha-n)} \left(\int_{B(x'',x_n/4)} dy\right)^{1/q} \\ &\qquad + M_6 x_n^{1+\beta/q} \left(\int_{E_2 - B(x'',x_n/4)} |x''-y|^{(\alpha-n-1)q}\, dy\right)^{1/q} \\ &\quad\leqq M_7 x_n^{\beta/q+\alpha-n+n/q}.\end{aligned}$$

Finally, in the same way, we obtain

$$\begin{aligned}
&\left(\int_{E_3} K_\alpha(x,y)^q |y_n|^\beta\, dy\right)^{1/q} \\
&\quad \leqq M_8 \left(\int_{B(x', x_n/4)} |x'-y|^{(\alpha-n)q} |y_n|^\beta\, dy\right)^{1/q} \\
&\qquad + M_8 x_n \left(\int_{R^n - B(x', x_n/4)} |x'-y|^{(\alpha-n-1)q} |y_n|^\beta\, dy\right)^{1/q} \\
&\quad \leqq M_9 x_n^{\alpha-n+(n+\beta)/q}.
\end{aligned}$$

The required inequality now follows.

LEMMA 3.3. *Let*

$$n-\alpha < \frac{n-1}{q} < n-\alpha+2.$$

If $x=(x', x_n) \in D$ *and* $y=(y', y_n) \in R^n$, *then*

$$\left(\int_{R^{n-1}} |K_\alpha(x,y)|^q\, dx'\right)^{1/q} \leqq M x_n |y_n - x_n/2| \big(x_n + |y_n|\big)^{\alpha-n-2+(n-1)/q}.$$

Proof. For $x=(x', x_n) \in D$ and $y=(y', y_n) \in R^n$, set

$$r=|x'-y'|, \qquad a=|y_n - x_n|, \qquad b=|y_n|.$$

If $y_n > x_n/2$, then $a<b$, so that

$$\begin{aligned}
&\left(\int_{R^{n-1}} K_\alpha(x,y)^q\, dx'\right)^{1/q} \\
&\quad \leqq M_1 x_n |y_n - x_n/2| \left(\int_0^\infty \left[(r+a)^{\alpha-n}(r+b)^{-2}\right]^q r^{n-2}\, dr\right)^{1/q} \\
&\quad \leqq M_1 x_n |y_n - x_n/2| \left(b^{-2q}\int_0^b r^{(\alpha-n)q} r^{n-2}\, dr + \int_b^\infty r^{(\alpha-n-2)q} r^{n-2}\, dr\right)^{1/q} \\
&\quad \leqq M_2 x_n |y_n - x_n/2| b^{\alpha-n-2+(n-1)/q}.
\end{aligned}$$

The similar result also holds for $y_n < x_n/2$.

THEOREM 3.1. *Let* $1 \leqq p \leqq q$, $\beta < p-1$,

$$\frac{n-\alpha p}{p(n-1)} < \frac{1}{q}$$

and

$$\frac{n-\alpha p+\beta}{p(n-1)} < \frac{1}{q} < \frac{n-(\alpha-1)p+\beta}{p(n-1)}.$$

If f is a nonnegative measurable function on R^n satisfying (1.2) *and* (1.3), *then*

$$\lim_{r\to 0} r^{(n-\alpha p+\beta)/p-(n-1)/q} S_q(v_r) = 0,$$

where $v_r(x') = U_\alpha f(x', r) - U_\alpha f(x', 0)$ for $r > 0$.

Proof. We treat only the case $p > 1$. Take (δ, γ) such that

$$\beta < \gamma < p(n-\alpha+1)\delta + \beta - p(n-1)/q, \tag{3.1}$$

$$p(n-\alpha+1)\delta + (\alpha-1)p - n < \gamma < p(n-\alpha)\delta + \alpha p - n, \tag{3.2}$$

$$\beta < \gamma < p-1,$$

$$\delta > \frac{n-\alpha p}{p(n-\alpha)}, \tag{3.3}$$

and

$$\frac{n-1}{q(n-\alpha+1)} < \delta < \frac{n-1}{q(n-\alpha)}. \tag{3.4}$$

As in the proof of Theorem 2.1, set $a = (1-\delta)p^*$ and $b = -\gamma p^*/p$. Then $b > -1$ and by (3.3),

$$\alpha > \frac{n}{a^*}, \qquad \frac{1}{a} + \frac{1}{a^*} = 1.$$

Further, (3.2) implies

$$\frac{n}{a^*} - \alpha < \frac{b}{a} < \frac{n}{a^*} - \alpha + 1.$$

Hence, using Hölder's inequality and Lemma 3.2, we have

$$\begin{aligned} \left| v_{x_n}(x') \right| &\leqq \left(\int \left| K_\alpha(x,y) \right|^{(1-\delta)p^*} |y_n|^{-\gamma p^*/p}\, dy \right)^{1/p^*} \\ &\quad \times \left(\int |K_\alpha(x,y)|^{\delta p} |y_n|^{\gamma} f(y)^p\, dy \right)^{1/p} \\ &\leqq M_1 x_n^{(\alpha-n)(1-\delta)+n/p^*-\gamma/p} \left(\int \left| K_\alpha(x,y) \right|^{\delta p} |y_n|^{\gamma} f(y)^p\, dy \right)^{1/p}. \end{aligned}$$

In view of Minkowski's inequality, we find

$$S_q(v_{x_n}) \leqq M_1 x_n^{(\alpha-n)(1-\delta)+n/p^*-\gamma/p}$$
$$\times \left\{ \int \left(\int_{R^{n-1}} |K_\alpha(x,y)|^{\delta q}\, dx' \right)^{p/q} |y_n|^\gamma f(y)^p\, dy \right\}^{1/p}.$$

Here, noting (3.4), we have by Lemma 3.3

$$\left(\int_{R^{n-1}} |K_\alpha(x,y)|^{\delta q}\, dx' \right)^{p/q}$$
$$\leqq M_2 \left[x_n |y_n - x_n/2| (x_n + |y_n|)^{\alpha-n-2+(n-1)/\delta q} \right]^{\delta p}.$$

Consequently,

$$S_q(v_r) \leqq M_3 r^{(\alpha-n)(1-\delta)+n/p^*-\gamma/p+\delta}$$
$$\times \left(\int \left\{ \left[|2y_n - r| (r + |y_n|)^{\alpha-n-2+(n-1)/\delta q} \right]^{\delta p} |y_n|^{\gamma-\beta} \right\} \right.$$
$$\left. \times |y_n|^\beta f(y)^p\, dy \right)^{1/p}.$$

It follows from (3.1) that

$$r^{(n-\alpha p+\beta)/p-(n-1)/q} r^{(\alpha-n)(1-\delta)+n/p^*-\gamma/p+\delta}$$
$$= r^{(n-\alpha+1)\delta+(\beta-\gamma)/p-(n-1)/q} \to 0$$

as $r \to 0$. Consider

$$K(r, y_n) = r^{p[(n-\alpha p+\beta)/p-(n-1)/q]} r^{p[(\alpha-n)(1-\delta)+n/p^*-\gamma/p]}$$
$$\times \left[\left[|2y_n - r| (r + |y_n|) \right]^{\alpha-n-2+(n-1)/\delta q} \right] \delta p |y_n|^{\gamma-\beta}.$$

Further, if $r < |y_n|$, then

$$K(r, y_n) \leqq M_4 (r/|y_n|)^{(n-\alpha+1)\delta p+(\beta-\gamma)-p(n-1)/q} \leqq M_4;$$

if $|y_n| \leqq r$, then

$$K(r, y_n) \leqq M_5 (|y_n|/r)^{\beta-\gamma} \leqq M_5.$$

Hence Lebesgue's dominated convergence theorem implies that

$$\lim_{r\to\infty} r^{(n-\alpha p+\beta)/p-(n-1)/q} S_q(v_r) = 0.$$

LEMMA 3.4. *Let $n-1 < q(n-\alpha)$. If $x = (x', x_n) \in D$, $y = (y', y_n)$, and $y_n > x_n/2$, then*

$$\left(\int_{R^{n-1}} |K_\alpha(x, y)|^q \, dx' \right)^{1/q} \leqq M x_n |y_n - x_n/2| \, |x_n - y_n|^{\alpha - n + (n-1)/q} (x_n + |y_n|)^{-2};$$

if $y_n \leqq x_n/2$, then

$$\left(\int_{R^{n-1}} |K_\alpha(x, y)|^q \, dx' \right)^{1/q} \leqq M x_n |y_n - x_n/2| \, |y_n|^{\alpha - n + (n-1)/q} (x_n + |y_n|)^{-2}.$$

Proof. For $x = (x', x_n)$ and $y = (y', y_n)$, set

$$r = |x' - y'|, \qquad a = |y_n - x_n|, \qquad b = |y_n|.$$

If $y_n > x_n/2$, then $a < b$, so that Lemma 3.1 gives

$$\begin{aligned} &\left(\int_{R^{n-1}} K_\alpha(x, y)^q \, dx' \right)^{1/q} \\ &\qquad \leqq M_1 x_n (y_n - x_n/2) \left(\int_0^\infty \left[(r+a)^{\alpha - n} (r+b)^{-2} \right]^q r^{n-2} \, dr \right)^{1/q} \\ &\qquad \leqq M_2 x_n (y_n - x_n/2) a^{\alpha - n + (n-1)/q} b^{-2}. \end{aligned}$$

The second case also follows in the same manner.

THEOREM 3.2. *Let $1 \leqq p \leqq q$, $\beta < p - 1$,*

$$\frac{n - \alpha p - 1}{p(n-1)} < \frac{1}{q} \leqq \frac{n - \alpha p}{p(n-1)},$$

and

$$\frac{n - \alpha p + \beta}{p(n-1)} < \frac{1}{q} < \frac{n - (\alpha - 1)p + \beta}{p(n-1)}.$$

If f is a nonnegative measurable function on R^n satisfying (1.2) *and* (1.3), *then*

$$\liminf_{r\to 0} r^{(n-\alpha p+\beta)/p-(n-1)/q} S_q(v_r) = 0,$$

where $v_r(x') = U_\alpha f(x', r) - U_\alpha f(x', 0)$ for $r > 0$.

Proof. We treat only the case $p > 1$. First note that

$$\frac{1}{q} \leqq \frac{n - \alpha p}{p(n-1)} < \frac{n-\alpha}{n-1}.$$

Take (δ, γ) such that

$$\beta < \gamma < p - 1, \qquad 0 < \delta < 1,$$

$$p(n-\alpha)\delta + \beta - p(n-1)/q < \gamma < p(n-\alpha+1)\delta + \beta - p(n-1)/q, \tag{3.5}$$

$$p(n-\alpha+1)\delta + (\alpha-1)p - n < \gamma < p(n-\alpha)\delta + \alpha p - n, \tag{3.6}$$

$$\delta > \frac{n-\alpha p}{p(n-\alpha)}, \tag{3.7}$$

and

$$\frac{n-1}{q(n-\alpha)} < \delta < \frac{n-1}{q(n-\alpha)} + \frac{1}{p(n-\alpha)}. \tag{3.8}$$

As before, set $a = (1-\delta)p^*$ and $b = -\gamma p^*/p$. Then $b > -1$, $\alpha > n/a^*$ by (3.7), and

$$\frac{n}{a^*} - \alpha < \frac{b}{a} < \frac{n}{a^*} - \alpha + 1$$

by (3.6). Applying Hölder's inequality and Lemma 3.2, we have

$$\begin{aligned} \left| v_{x_n}(x') \right| &\leqq \left(\int \left| K_\alpha(x,y) \right|^{(1-\delta)p^*} |y_n|^{-\gamma p^*/p}\, dy \right)^{1/p^*} \\ &\quad \times \left(\int \left| K_\alpha(x,y) \right|^{\delta p} |y_n|^{\gamma} f(y)^p\, dy \right)^{1/p} \\ &\leqq M_1 x_n^{(\alpha-n)(1-\delta)+n/p^*-\gamma/p} \left(\int \left| K_\alpha(x,y) \right|^{\delta p} |y_n|^{\gamma} f(y)^p\, dy \right)^{1/p}. \end{aligned}$$

Minkowski's inequality yields

$$\begin{aligned} S_q(v_{x_n}) &\leqq M_1 x_n^{(\alpha-n)(1-\delta)+n/p^*-\gamma/p} \\ &\quad \times \left\{ \int \left(\int_{R^{n-1}} \left| K_\alpha(x,y) \right|^{\delta q} dx' \right)^{p/q} |y_n|^{\gamma} f(y)^p\, dy \right\}^{1/p}. \end{aligned}$$

Here, in view of Lemma 3.4, we find

$$\left(\int_{R^{n-1}} |K_\alpha(x,y)|^{\delta q}\,dx'\right)^{p/q}$$
$$\leqq M_2\left[x_n|y_n - x_n/2|\,|x_n - y_n|^{\alpha-n+(n-1)/\delta q}(x_n + |y_n|)^{-2}\right]^{\delta p}$$
$$+ M_2\left[x_n|y_n - x_n/2|\,|y_n|^{\alpha-n+(n-1)/\delta q}(x_n + |y_n|)^{-2}\right]^{\delta p}.$$

Consequently,

$$S_q(v_r) \leqq M_3 r^{(\alpha-n)(1-\delta)+n/p^*-\gamma/p+\delta}$$
$$\times\left\{\int\left(\left[|2y_n - r|\,|r - y_n|^{\alpha-n+(n-1)/\delta q}\right.\right.\right.$$
$$\left.\left.\times(r+|y_n|)^{-2}\right]^{\delta p}|y_n|^{\gamma-\beta}\right)|y_n|^{\beta}f(y)^p\,dy$$
$$+\int\left(\left[|2y_n - r|\,|y_n|^{\alpha-n+(n-1)/\delta q}(r+|y_n|)^{-2}\right]^{\delta p}\right.$$
$$\left.\left.\times|y_n|^{\gamma-\beta}\right)|y_n|^{\beta}f(y)^p\,dy\right\}^{1/p}.$$

Here note from (3.5) that

$$r^{(n-\alpha p+\beta)/p-(n-1)/q}r^{(\alpha-n)(1-\delta)+n/p^*-\gamma/p+\delta}$$
$$= r^{(n-\alpha+1)\delta+(\beta-\gamma)/p-(n-1)/q} \to 0$$

as $r \to 0$. Consider

$$k(r, y_n) = r^{p[(n-\alpha p+\beta)/p-(n-1)/q]}r^{p[(\alpha-n)(1-\delta)+n/p^*-\gamma/p+\delta]}$$
$$\times\left[|2y_n - r|\left\{|r-y_n|^{\alpha-n+(n-1)/\delta q} + |y_n|^{\alpha-n+(n-1)/\delta q}\right\}\right.$$
$$\left.\times(x_n+|y_n|)^{-2}\right]^{\delta p}|y_n|^{\gamma-\beta}.$$

If $3r/2 < y_n$, then

$$k(r, y_n) \leqq M_4(r/|y_n|)^{(n-\alpha+1)\delta p+(\beta-\gamma)-p(n-1)/q} \leqq M_5;$$

if $-3r/2 < y_n \leqq r/2$, then

$$k(r, y_n) \leqq M_6\left\{1 + (|y_n|/r)^{(\alpha-n)p\delta+p(n-1)/q+\gamma-\beta}\right\} \leqq M_7;$$

finally, if $|r - y_n| < r/2$, then

$$k(r, y_n) \leqq M_8 r^d|r-y_n|^{-d}$$

with $d = (n-\alpha)p\delta - p(n-1)/q$. Here note that $0 < d < 1$ by (3.8). Hence it follows from Lemma 2.3 that

$$\liminf_{r \to 0} r^{(n-\alpha p+\beta)/p-(n-1)/q} S_q(v_r) = 0.$$

4. HYPERPLANE MEANS OF HARMONIC FUNCTIONS

First we note the following integral representation of p-precise functions on the half space D by [8, Lemma 3].

LEMMA 4.1. *If $-1 < \beta < p-1$ and u is a p-precise function on D satisfying*

$$\int_D |\operatorname{grad} u(x)|^p x_n^\beta \, dx < \infty, \tag{4.1}$$

then there exists a constant c such that

$$u(x) = c \sum_{j=1}^{n} \int \frac{x_j - y_j}{|x-y|^n} \frac{\partial \bar{u}}{\partial y_j} \, dy + A \tag{4.2}$$

for almost every $x \in D$, where A is a constant determined by u, $\bar{u}(y', y_n) = u(y', y_n)$ for $y_n > 0$, and $\bar{u}(y', y_n) = u(y', -y_n)$ for $y_n < 0$.

Remark 4.1. If in addition u is harmonic in D, then (4.2) holds for every $x \in D$ and, moreover,

$$\sum_{j=1}^{n} \int_{B(x,r)} \frac{x_j - y_j}{|x-y|^n} \frac{\partial \bar{u}}{\partial y_j} \, dy = 0$$

whenever $0 < r < x_n$.

Remark 4.2. Note that

$$\int_{R^n} |\operatorname{grad} \bar{u}(x)|^p |x_n|^\beta \, dx < \infty.$$

Consider

$$k_j(x, y) = \frac{x_j - y_j}{|x-y|^n} \qquad \text{for } y \in R^n - B(x, x_n);$$

set $k_j(x, y) = 0$ on $B(x, x_n)$.

LEMMA 4.2. *If* $-1 < \beta < (n-1)q - n$, *then*

$$\left(\int \left| k_j(x,y) \right|^q |y_n|^\beta \, dy \right)^{1/q} \leqq M x_n^{1-n+(\beta+n)/q}$$

for any $x = (x', x_n) \in D$.

Proof. As in Lemma 2.1, we have

$$\begin{aligned} \left(\int_{E_1} \left| k_j(x,y) \right|^q |y_n|^\beta \, dy \right)^{1/q} & \\ &\leqq M_1 \left(\int_{E_1 - B(x, x_n)} |x-y|^{q(1-n)} \left(|x_n| + |y_n - x_n| \right)^\beta \, dy \right)^{1/q} \\ &\leqq M_2 \left(\int_{x_n}^{\infty} r^{q(1-n)+\beta} r^{n-1} \, dr \right)^{1/q} \\ &\leqq M_3 x_n^{1-n+(n+\beta)/q} \end{aligned}$$

and

$$\begin{aligned} \left(\int_{E_2} \left| k_j(x,y) \right|^q |y_n|^\beta \, dy \right)^{1/q} &\leqq M_4 \left(\int_{E_2} \left(|x' - y| + x_n \right)^{q(1-n)} |y_n|^\beta \, dy \right)^{1/q} \\ &\leqq M_5 \left(\int_0^{\infty} (r + x_n)^{q(1-n)} r^\beta r^{n-1} \, dr \right)^{1/q} \\ &\leqq M_6 x_n^{1-n+(\beta+n)/q}. \end{aligned}$$

LEMMA 4.3. *Let* $q > 1$. *For* $x = (x', x_n)$ *and* $y = (y', y_n)$ *in* R^n,

$$\left(\int_{R^{n-1}} \left| k_j(x,y) \right|^q \, dx' \right)^{1/q} \leqq M \left(x_n + |x_n - y_n| \right)^{-(n-1)/q^*},$$

where $1/q + 1/q^* = 1$.

Proof. If $|x_n - y_n| < x_n/2$ and $|x-y| > x_n$, then $x_n^2 < |x' - y'|^2 + (x_n - y_n)^2 < |x' - y'|^2 + (x_n/2)^2$, so that $x_n/2 < |x' - y'|$. Thus we have

$$\begin{aligned} \left(\int \left| k_j(x,y) \right|^q \, dx' \right)^{1/q} &\leqq M_1 \left(\int_0^{\infty} (r + x_n)^{q(1-n)} r^{n-2} \, dr \right)^{1/q} \\ &\leqq M_2 x_n^{1-n+(n-1)/q}. \end{aligned}$$

In general we have

$$\left(\int \left|k_j(x,y)\right|^q dx'\right)^{1/q} \leqq M_3\left(\int_0^\infty \left(r^2+(x_n-y_n)^2\right)^{q(1-n)/2} r^{n-2}\,dr\right)^{1/q}$$
$$\leqq M_4 |x_n - y_n|^{1-n+(n-1)/q}.$$

Therefore the required inequality follows.

THEOREM 4.1. *Let* $-1 < \beta < p-1$, $1 \leqq p \leqq q$, *and*

$$\frac{1}{q} < \frac{n-p+\beta}{p(n-1)}.$$

If u is a harmonic function on D satisfying (4.1), *then there exists a number A such that*

$$\lim_{r\to 0} r^{(n-p+\beta)/p-(n-1)/q} S_q(u_r - A) = 0,$$

where $u_r(x') = u(x', r)$ *for* $r > 0$.

Proof. We are concerned only with the case $p > 1$. First note that

$$\frac{1}{q} < \frac{n-p+\beta}{p(n-1)} < 1.$$

Take (δ, γ) such that

$$\beta < \gamma < p-1, \qquad 1/q < \delta < 1$$

and

$$p(n-1)\delta + p - n < \gamma < p(n-1)\delta + \beta - p(n-1)/q. \quad (4.3)$$

Set $a = (1-\delta)p^*$ and $b = -\gamma p^*/p$, where $1/p + 1/p^* = 1$. Then note that $b > -1$ and by the left inequality of (4.3),

$$\frac{b}{a} < \frac{n}{a^*} - 1,$$

where $1/a + 1/a^* = 1$. By Lemma 4.1, u is of the form

$$u(x) = c\sum_{j=1}^{n} \int \frac{x_j - y_j}{|x-y|^n}\,\frac{\partial \bar{u}}{\partial y_j}\,dy + A.$$

Hence, using Hölder's inequality and Lemma 4.2, we have

$$\begin{aligned}
&\left|u_{x_n}(x') - A\right| \\
&\qquad \leqq |c| \sum_{j=1}^{n} \left(\int \left| k_j(x,y) \right|^{(1-\delta)p^*} |y_n|^{-\gamma p^*/p} \, dy \right)^{1/p^*} \\
&\qquad\qquad \times \left(\int \left| k_j(x,y) \right|^{\delta p} |y_n|^{\gamma} f(y)^p \, dy \right)^{1/p} \\
&\qquad \leqq M_1 x_n^{(1-n)(1-\delta)+n/p^*-\gamma/p} \sum_{j=1}^{n} \left(\int \left| k_j(x,y) \right|^{\delta p} |y_n|^{\gamma} f(y)^p \, dy \right)^{1/p},
\end{aligned}$$

where $f(y) = |\mathrm{grad}\, \bar{u}(y)|$. Note here that f satisfies (1.2). In view of Minkowski's inequality we obtain

$$\begin{aligned}
S_q\left(u_{x_n} - A\right) &\leqq M_1 x_n^{(1-n)(1-\delta)+n/p^*-\gamma/p} \\
&\qquad \times \sum_{j=1}^{n} \left\{ \int \left(\int_{R^{n-1}} \left| k_j(x,y) \right|^{\delta q} dx' \right)^{p/q} |y_n|^{\gamma} f(y)^p \, dy \right\}^{1/p}.
\end{aligned}$$

By Lemma 4.3, we find

$$\begin{aligned}
&S_q(u_r - A) \\
&\quad \leqq M_2 r^{(1-n)(1-\delta)+n/p^*-\gamma/p} \\
&\qquad \times \left(\int \left\{ \left[(r + |y_n - r|)^{(1-n)+(n-1)/\delta q} \right]^{\delta p} |y_n|^{\gamma-\beta} \right\} |y_n|^{\beta} f(y)^p \, dy \right)^{1/p}.
\end{aligned}$$

Consider

$$\begin{aligned}
K(r, y_n) &= r^{p[(n-p+\beta)/p-(n-1)/q]} r^{p[(1-n)(1-\delta)+n/p^*-\gamma/p]} \\
&\qquad \times (r + |r - y_n|)^{[1-n+(n-1)/\delta q]\delta p} |y_n|^{\gamma-\beta}.
\end{aligned}$$

In view of the right inequality of (4.3), we see that

$$\lim_{r\to 0} K(r, y_n) = 0$$

for fixed y_n. Further, if $|y_n| > 2r$, then

$$K(r, y_n) \leqq M_3\left(r/|y_n|\right)^{(n-1)\delta p+(\beta-\gamma)-p(n-1)/q} \leqq M_3;$$

if $|y_n| \leqq 2r$, then

$$K(r, y_n) \leqq M_4.$$

Hence it follows from Lebesgue's dominated convergence theorem that

$$\liminf_{r\to 0} r^{(n-p+\beta)/p-(n-1)/q} S_q(u_r - A) = 0.$$

Remark 4.3. In case $p > 1$, $-1 < \beta < p-1$, and u is a harmonic function on D satisfying (4.1), it follows from Theorem 2.2 that $S_q(u_r - A)$ is bounded on $(0, \infty)$ for $1/q = (n-p+\beta)/p(n-1) > 0$. Professor Yamashita [15, Theorem 3] treated the case when u is a harmonic function on B satisfying

$$\int_B |\operatorname{grad} u(x)|^2 (1-|x|^2)^\beta \, dx < \infty, \qquad 0 \leqq \beta \leqq 1.$$

The case $\beta = p-1$ is open.

5. BEST POSSIBILITIES

We show that the power of r in Theorem 2.1 is best possible. For this purpose, consider the function

$$f(y) = |y_n|^a |y|^{-b}, \qquad \text{for } y \in B(0,1) - D;$$

set $f(y) = 0$ elsewhere. For $\delta > 0$, let

$$a = -(\beta+1)/p + \delta, \qquad b = (n-1)/p.$$

Then it is easy to see that

$$\int f(y)^p |y_n|^\beta < \infty.$$

Define a harmonic function u on D by setting

$$u(x) = \int \frac{x_n - y_n}{|x-y|^n} f(y) \, dy.$$

In view of [8, Lemma 1], if $-1 < \beta < p-1$ and $p > 1$, then

$$\int_{R^n} |\operatorname{grad} u(x)|^p |x_n|^\beta \, dx < \infty.$$

For $x \in D \cap B(0, 1/2)$, find

$$u(x) \geqq \int_{B(\bar{x}, x_n/2)} \frac{x_n - y_n}{|x-y|^n} f(y)\, dy$$
$$\geqq M_1 x_n^{1+a} |x|^{-b},$$

where $\bar{x} = (x', -x_n)$ for $x = (x', x_n)$. Hence, if $q > 0$ and $0 < r < 1/4$, then we have

$$S_q(u_r) \geqq M_1 r^{1+a} \left(\int_{\{x' : |x'| < r\}} \left(|x'|^2 + r^2\right)^{-bq/2} dx' \right)^{1/q}$$
$$\geqq M_2 r^{1+a-b+(n-1)/q}$$
$$= M_2 r^{-(n-p+\beta)/p+(n-1)/q+\delta}.$$

This implies that the exponent of Theorem 2.1 is best possible. Similarly, if we set

$$v(x) = \int \left(|x-y|^{1-n} - |\bar{x} - y|^{1-n} \right) f(y)\, dy,$$

then

$$S_q(v_r) \geqq M_3 r^{-(n-p+\beta)/p+(n-1)/q+\delta}.$$

Thus the exponent of Theorems 3.1 and 3.2 is best possible, too.

REFERENCES

1. S. J. Gardiner, Growth properties of pth means of potentials in the unit ball, *Proc. Amer. Math. Soc.* **103** (1988), 861–869.
2. S. J. Gardiner, Local growth properties of superharmonic functions, *Arch. Math.* **54** (1990), 52–60.
3. Y. Mizuta, Integral representations of Beppo Levi functions of higher order, *Hiroshima Math. J.* **4** (1974), 375–396.
4. Y. Mizuta, Existence of various boundary limits of Beppo Levi functions of higher order, *Hiroshima Math. J.* **9** (1979), 717–745.
5. Y. Mizuta, Minimally semi-fine limits of Green potentials of general order, *Hiroshima Math. J.* **12** (1982), 505–511.
6. Y. Mizuta, On the boundary behavior of superharmonic functions in a half space, *J. Math. Soc. Japan* **34** (1982), 269–278.
7. Y. Mizuta, On the growth of α-potentials in R^n and thinness of sets, *Hiroshima Math. J.* **15** (1985), 585–599.
8. Y. Mizuta, Boundary behavior of p-precise functions on a half space of R^n, *Hiroshima Math. J.* **18** (1988), 73–94.

9. Y. Mizuta, Integral representations of Beppo Levi functions and the existence of limits at infinity, *Hiroshima Math. J.* **19** (1989), 259–279.
10. Y. Mizuta, Growth properties of p-th means of Green potentials in a half space, *J. Math. Soc. Japan* **44** (1992), 1–12.
11. Y. Mizuta, Continuity properties of potentials and Beppo-Levi-Deny functions, *Hiroshima Math. J.* **23** (1993), 79–153.
12. M. Ohtsuka, Extremal length and precise functions in 3-space, lecture notes, Hiroshima University, 1973.
13. M. Stoll, Rate of growth of pth means of invariant potential in the unit ball of C^n, *J. Math. Anal. Appl.* **143** (1989), 480–499.
14. M. Stoll, Rate of growth of pth means of invariant potentials in the unit ball of C^n, II, *J. Math. Anal. Appl.* **165** (1992), 374–398.
15. S. Yamashita, Dirichlet-finite functions and harmonic functions, *Illinois J. Math.* **25** (1981), 626–631.
16. W. P. Ziemer, Extremal length as a capacity, *Michigan Math. J.* **17** (1970), 117–128.